Ordinary Differential Equations
Exercise Sheet 4

Exercise 1. *Study the phase space for the systems* $\vec{y}' = Ay$, *where*

$$(\mathrm{i})\quad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad (\mathrm{ii})\quad A = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}, \qquad (\mathrm{iii})\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Exercise 2. *Consider the system* $\vec{y}' = Ay$, *where*

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Find the stable manifold, ie. the points $\vec{y}(0)$ *for which the corresponding trajectories tend to 0, as* $t \to +\infty$.

Exercise 3. *Consider the following system:*

$$\begin{cases} y_1' = y_2 - y_1(1 - y_1^2 - y_2^2) \\ y_2' = -y_1 - y_2(1 - y_1^2 - y_2^2) \end{cases}$$

(i) *Show that 0 is the only equilibrium point and find its type.*

(ii) *Show that the open unit disk* $D = \{y_1^2 + y_2^2 < 1\}$ *is invariant under the system, ie. for* $(y_1(0), y_2(0)) \in D$ *it holds* $(y_1(t), y_2(t)) \in D$ *for all* $t > 0$.

(iii) *Show that all solutions with initial conditions in* D *converge to 0 exponentially, as* $t \to +\infty$.

Exercise 4. *Consider the system:*

$$\begin{cases} y_1' = y_1^3 - y_1 + \frac{1}{\sqrt{n}} y_2^2 \\ y_2' = y_2(y_1^2 - \frac{1}{\sqrt{n}} y_1) \end{cases}$$

where $n \geq 2$.

(i) *Find all equilibrium points and their type.*

(ii) *Show that trajectories with* $y_1^2(0) + y_2^2(0) < 1$ *satisfy* $y_1^2(t) + y_2^2(t) < 1$ *for all* $t > 0$ *and that they tend to 0 exponentially, as* $t \to +\infty$.

(iii) *Show that trajectories with* $y_1^2(0) + y_2^2(0) = 1$ *stay on the unit circle, ie.* $y_1^2(t) + y^2(t) = 1$ *for all* $t > 0$ *and study their convergence, as* $t \to +\infty$.

Exercise 5. *Show that all solutions to the equation*

$$y'' + (y')^5 + y + y^4 = 0$$

tend to 0, as $t \to +\infty$.

Exercise 6. *Write the equation*

$$y'' + (1 - e^{-y^2})y = 0$$

as a first order system and draw its trajectories in phase space.

Exercise 7. *Consider the system*

$$\begin{cases} y_1' = f_1(y_1, y_2) \\ y_2' = f_2(y_1, y_2) \end{cases}$$

where f_1, f_2 *are continuous functions in* $\mathbb{R}^2$. *Assume there is* $k > 0$ *such that* $y_1 f_1(y_1, y_2) + y_2 f_2(y_1, y_2) < 0$, *for all* $y_1^2 + y_2^2 > k$. *Then show that all solutions to the previous system are bounded for* $t \geq 0$.